WILD COOL MATH!

CURIOUS MATHEMATICS FOR FUN AND JOY

MAY 2013

PROMOTIONAL CORNER:
Have you an event, a workshop, a website, some materials you would like to share with the world? Let me know! If the work is about deep and joyous and real mathematical doing I would be delighted to mention it here.

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Are you in the DC/Maryland are looking for an exciting, rich and challenging summer-course experience for students?
The Institute for Academic Challenge is offering a really cool course on Combinatorics, August 12-23. Check out www.academicchallenge.org/classes.html.

PUZZLER: Here's a delight for dividing by nine. To compute $12401 \div 9$, for example, just read form left to right and record partial sums. The partial sums give the quotient and the remainder!

1	= 1
1+2	= 3
1+2+4	= 7
1+2+4+0	= 7
1+2+4+0+1	= 8

12401 ÷ 9 = 1377 remainder 8

The puzzle is: Why does this work?

COMMENT: If we didn't carry digits in our arithmetic system, this method would always hold. For instance,

$$3282 \div 9 = 3 \mid 5 \mid 13 \ remainder\ 15$$

is mathematically correct!

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DIVIDING BY 97 AND OTHER SUCH QUANTITIES

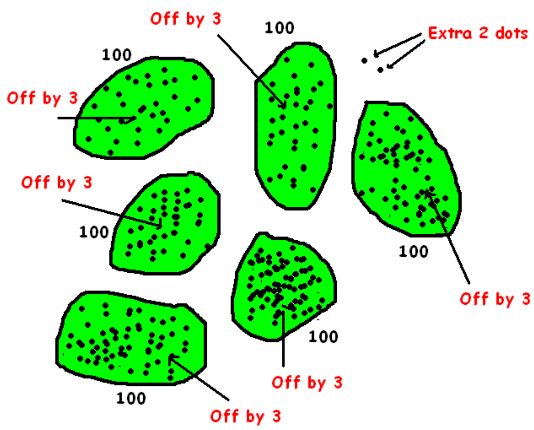
Quick! What's $602 \div 97$*?*

Well…

602 is basically 600,
97 is essentially 100, and
$600 \div 100 = 6$.

So $602 \div 97$ is basically 6. How far off is this answer?

A picture of 602 dots shows that each group of 100 is off from being a group of 97 by three - and there are two dots ignored.



Thus $602 \div 97$ equals 6 with a remainder of six groups of the three plus two, namely 20, giving

$$602 \div 97 = 6\frac{20}{97},$$

This has value a smidgeon over 6.2.

In the same way:

$718 \div 99$ equals 7 with a remainder of $7 \times 1 + 18 = 25$ (so $\frac{718}{99} = 7\frac{25}{99} \approx 7.25$),

$4142 \div 996$ equals 4 with a remainder of $4 \times 4 + 142 = 158$ (so

$$\frac{4142}{96} = 4\frac{158}{996} \approx 4.16),$$

and

$1378 \div 98$ equals 13 with a remainder of $26 + 78 = 104$! That is,

$$1378 \div 98 = 14\frac{6}{98}.$$

Query: Look at the picture on the left. Can you use it to see that $602 \div 103$ equals 6 with a remainder of -16, a negative remainder? So $602 \div 103 = 6\frac{-16}{103}$. What is approximate decimal value of this?

What is $598 \div 103$?

COMMENT**:** If $x = y$ and $z = y$, then surely it is true that $x = z$. Right?

Many school curricular have students write, for example,

$$29 \div 8 = 3\,R\,5$$

to mean "29 divided by 8 is 3 with a remainder of 5" or

$$32 \div 9 = 3\,R\,5$$

to mean "32 divided by 9 is 3 with a remainder of 5."

Super. It looks like we just proved $29 \div 8 = 32 \div 9$!

Query:
A good way to divide a number by 5 is to double it and divide the result by 10.

Along this line… Is it worth observing that 95 is a multiple of 19?

For example, to estimate $\frac{70}{19}$ we can estimate $\frac{350}{95}$ instead (and see it is $3\frac{65}{95}$ with value a smidgeon larger than 3.65.)

Is it worth noting that 94 is a multiple of 47? Or that 102 is a multiple of 17? And so on.

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DIVISIBILITY RULES

One whole number is said to be *divisible* by another whole number k if it leaves no remainder upon division by k.

There are many divisibility rules for different values of k. Are you familiar with the following?

Divisibility by 2: *A number is divisible by two only if its final digit is 0, 2, 4, 6, or 8.*

Divisibility by 5: *A number is divisible by five only if its final digit is 0 or 5.*

Divisibility by 10: *A number is divisible by ten only if its final digit is 0.*

Divisibility by 1:
All numbers are divisible by 1.

How about …?

Divisibility by 4: *A number is divisible by four only if its final two digits represent a multiple of four (that is, a number that can be halved twice).*

Divisibility by 8: *A number is divisible by eight only if its three final digits represent a number that can be halved thrice.*

Divisibility by 16: *A number is divisible by* 16 *only if its four final digits represent a number that can be halved four times.*

Query: And the divisibility rules for 32, and 64, and 128, and so on are…?

Divisibility by 3 or 9: *A number is divisible by three/nine only if the sum of its digits is a multiple of three/nine.*

Divisibility by 11: *A number is divisible by 11 only if the alternating sum of its digits is a multiple of 11.*

For example, 81709364 has alternating sum $8-1+7-0+9-3+6-4=22$, a multiple of 11, revealing that 81709364 is also a multiple of eleven!

Less well known …

Divisibility by 7: *Remove the last digit from the given number and subtract twice that digit from the number that remains. The original number is divisible by 7 only if the result of this operation is divisible by seven.*

For example, to test whether or not 68978 is divisibly by seven, remove the final "8" and subtract its double 16 from 6897, the number that remains. This gives $6897-16=6881$.

We can test whether or not 6881 is a multiple of seven the same way:

$$6881 \rightarrow 688-2=686$$

and once more

$$686 \rightarrow 68-12=56.$$

That the final result 56 is divisible by seven assures us that 68978 is also a multiple of seven.

Divisibility by 13: *Delete the final digit from a number and subtract 9 times that digit from the number that remains. The original number is divisible by 13 only if the result is divisible by 13.*
OR
Delete the final digit from a number and add 4 times that digit to the number that remains. The original number is divisible by 13 only if the result is divisible by 13.

Divisibility by 17: *Similar, but this time subtract 5 times the final digit OR instead add 12 times that final digit to what remains.*

Divisibility by 23: *Similar, but now add 7 times the final digit.*

Divisibility by 37: *Subtract 11 times the final digit.*

Divisibility by 101: *Subtract 10 times the final digit.*

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Divisibility rules can be weirder!

Divisibility by 11 again:
Divide the number under consideration into blocks of two and sum the blocks. If the sum is divisible by eleven, then so was the original number.

For example, to test whether or not 876535 is divisible by 11, compute:

$$87+65+35=187$$

To see if 187 is divisible by 11, think of it as 0187 and compute $01+87=88$. As 88 is a multiple of 11, so then is 187 and thus also 876535.

Divisibility by 37:
Delete the first digit of the number and add it to what remains three places in. The original number is divisible by 37 only if this new number is divisible by 37.

For example, consider the number 4579638. Deleting the first digit and "adding it three places in" gives:

$$\begin{array}{r} 579638 \\ +\quad 4 \\ \hline 583638 \end{array}$$

Repeating:

$$\begin{array}{r} 83638 \\ +\quad 5 \\ \hline 84138 \end{array}$$

$$\rightarrow \quad \begin{array}{r} 4138 \\ +\quad 8 \\ \hline 4218 \end{array}$$

$$\rightarrow \quad \begin{array}{r} 218 \\ +\quad 4 \\ \hline 222 \end{array}$$

Since 222 is a multiple of 37 (it is 6×37) we have that 4579638 is a multiple of 37.

CAN YOU EXPLAIN WHY THESE RULES WORK?

SPOILERS!

Don't read on if you want to figure out these rules for yourself.

I only have four techniques up my sleeve for deriving and creating divisibility rules.

1. Observe that every number is a multiple of ten (hundred, thousand, …) plus "stuff."

All numbers can be written in the form:

$$10N+b$$

that is, as a multiple of ten plus a single digit. (For example, $836=83\times 10+6$ and $5430=543\times 10+0$.)

Notice:

$$\frac{10N+b}{2}=5N+\frac{b}{2}$$

$$\frac{10N+b}{5}=2N+\frac{b}{5}$$

$$\frac{10N+b}{10}=N+\frac{b}{10}$$

We have that $b/2$ is a whole number only if $b=0,2,4,6$ or 8, thus explaining the divisibility rule for 2. And $b/5$ is a whole number only if $b=0$ or 5, and $b/10$ is a whole number only if $b=0$. These explain the divisibility rules for 5 and 10.

Any number can also be written in the form:

$$100N+bb$$

by which I mean a multiple of 100 plus a two-digit number. (For example, $836=8\times 100+36$ and $7=0\times 100+7$.)

Notice:

$$\frac{100N+bb}{4}=25N+\frac{bb}{4}.$$

This explains the divisibility rule for the number 4.

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Challenge: As 10 is a multiple of 2, we have that $1000 = 10 \times 10 \times 10$ is a multiple of $2 \times 2 \times 2 = 8$. Use this to explain the divisibility rule for 8.
(Notice, in general, 10^N a multiple of 2^N.)

Challenge: Explain the following divisibility rule for the number 25:
A number is divisible by 25 *only is its final two digits represent a number that is a multiple of* 25*, namely,* $00, 25, 50$ *or* 75.

2. Notice when powers of ten leave nice remainders.

Observe:

1	= 0 + 1
10	= 9 + 1
100	= 99 + 1
1000	= 999 + 1
10000	= 9999 + 1
etc	

Each power of ten is one more than a multiple of nine.

Since 1000, for instance, leaves a remainder of 1 when divided by nine, we have that $2000 = 1000 + 1000$ leaves a remainder of $1 + 1 = 2$ upon division by nine. And $4000 = 1000 + 1000 + 1000 + 1000$ leaves a remainder of $1 + 1 + 1 + 1 = 4$, and $9000 = 9 \times 1000$ leaves a remainder of 9 (which is the same as a remainder of zero).

Similarly, 500 leaves a remainder of $1+1+1+1+1 = 5$ and 70000000000 a remainder of 7.

A number such as 3627314 can be written

$$3000000 + 600000 + 20000 + 7000 + 300 + 10 + 4$$

and so leaves the remainder $3+6+2+7+3+1+4 = 26$ upon division by nine, which is the same as a remainder of 8 (and this is the sum of the digits $2+6$, of course!)

This idea establishes a strong result:

Upon division by nine, a number leaves the same remainder as does the sum of its digits.

Consequently:

A number is divisible by nine if its sum of digits is a multiple of nine.

Question: What is the divisibility rule for 6 for numbers written in base seven?

CASTING OUT NINES:
We have just seen that the a number leaves the same remainder upon division by nine as the sum of its digits does upon division by nine. Thus, for instance, 1,229,354,827 leaves a remainder equivalent to that of 1 + 2 + 2 + 9 + 3 + 4 + 5 + 8 + 2 + 7 = 43 when divided by nine. This corresponds to a remainder of 4 + 3 = 7.

One can simplify this process of adding digits by "casting out," that is, ignoring, any digit 9 that appears in the sum – it won't contribute to the remainder upon division by nine – and any combination of digits that sum to 9 (such as the 5 and the 4, and the final 2 and the 7, and a 1 and an 8). The sum of digits that survive must give the same remainder upon division by nine:

$$\cancel{1}\ 2\ 2\ \cancel{9}\ 3\ \cancel{4}\ \cancel{5}\ \cancel{8}\ \cancel{2}\ \cancel{7} \equiv 2+2+3=7$$

Practice: Cast out nines from the number 4504673542 to see that it leaves a remainder of 4 upon division by nine.

One can also cast out multiples of nine. For example, casting out $6+7+5$ from 6715108 (along with 1 and 8) shows that this number leaves a remainder of 1 upon division by nine.

Casting out nines can be used check arithmetical work. For example, we can readily see that the computation $547 \times 128 = 387,206$ must be incorrect: A quantity that leaves a remainder of 7 upon division by nine multiplied by one that leaves a remainder of 2 does not balance with an answer that gives a remainder of 8 upon division by nine. ($7 \times 2 = 14$

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corresponds to a remainder of 5, not 8.)

Exercise: Show, quickly, that the following sum is incorrect:

$$\begin{array}{r} 5478 \\ +\ 461 \\ +\ 1091 \\ +\ 2727 \\ +\ 6301 \\ \hline =\ 158358 \end{array}$$

There is, of course, a chance that remainders on erroneous computations happen to match. The method of casting out nines fails to detect those errors. (About how often might this occur?)

Exercise: Is the following computation correct?

$$8413 \times 259 \times 6547 = 14175696949$$

In past centuries banks and accounting firms used the method of casting out nines to check the arithmetic in their legers. A majority of errors were detected by it.

Theory Question: If N leaves a remainder r_1 upon division by nine, and M a remainder of r_2, is $N+M$ sure to leave a remainder of $r_1 + r_2$? And is $N \times M$ sure to leave a remainder of $r_1 \times r_2$? The casting out nines method assumes these results.

Question: The table

1	= 0 + 1
10	= 9 + 1
100	= 99 + 1
1000	= 999 + 1
10000	= 9999 + 1
etc	

also shows that each power of ten is one more than a multiple of three. Use this to explain the divisibility rule for the number three. (Do we actually have a stronger result?)

The powers of ten also leave nice remainders upon division by 11.

As 10 is one less than a multiple of eleven, so we could say that 10 leaves a remainder of -1 upon division by eleven.

Consequently, $100 = 10^2$ leaves a remainder of $(-1)^2 = 1$ upon division by eleven. (Indeed 100 is one more 9×11).

And $1000 = 10^3$ leaves a remainder of $(-1)^3 = -1$. (We have $1001 = 91 \times 11$.)

And $10000 = 10^4$ leaves a remainder of $(-1)^4 = 1$.

And so on.

Consequently, the number 83546, for instance, which can be written:

$$8 \times 10^4 + 3 \times 10^3 + 5 \times 10^2 + 4 \times 10 + 6$$

leaves a remainder equivalent to:

$$\begin{aligned} & 8 \times (-1)^4 + 3 \times (-1)^3 + 5 \times (-1)^2 + 4 \times (-1) + 6 \\ = \quad & 8 - 3 + 5 - 4 + 6 \end{aligned}$$

upon division by 11.

This work shows:

A number, upon division by 11, leaves the same remainder as does the alternating sum of its digits. (Provided the final digit is assigned a plus sign.)

Consequently:

A number is divisible by 11 precisely when the alternating sum of its digits is a multiple of 11, irrespective of the sign (plus or minus) of the final term.

(Explain the second part of the comment.)

Challenge: Describe a divisibility rule for the number 13 for numbers written in base twelve.

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Challenge:
A number leaves the same remainder upon division by 7 *as the sum of its digits modified by powers of three.*

For example, 2154 leaves the same remainder as does

$$2\times\underline{\underline{27}}+1\times\underline{\underline{9}}+5\times\underline{\underline{3}}+4\times\underline{\underline{1}}=82$$

which leaves the same remainder as

$$8\times\underline{\underline{3}}+2\times\underline{\underline{1}}=26$$

which leaves the same remainder as

$$2\times\underline{\underline{3}}+6\times\underline{\underline{1}}=12$$

which leaves the same remainder as

$$1\times\underline{\underline{3}}+2\times\underline{\underline{1}}=5.$$

Explain this (not very helpful) observation.

What is the corresponding result for testing divisibility by 13 (in base ten)?

We can also work with powers of 100.

As every number can be expressed in terms of powers of 100 (5672910, for instance, is $5\times 100^3+67\times 100^2+29\times 100+10\times 1$) and 100 is one more than a multiple of 11, we have:

Upon division by 11 ***a number leaves the same remainder as the sum of the "two-digit blocks" of that number.***

So 5672910 has a remainder of $05+67+29+10=111$ upon division by 11, which corresponds to a remainder of $01+11=12$, which is the same as a remainder of one.

3. Find multiples that are off by one from a multiple of ten.

Again write each number in the form:

$$10N+b$$

that is, as a multiple of ten plus a single digit. Let's consider divisibility by the number seven.

Since 21 is a multiple of seven we have that $\frac{10N+b}{7}$ is a whole number precisely when $\frac{10N+b-21b}{7}$ is a whole number.

Now

$$\frac{10N+b-21b}{7}=\frac{10\left(N-2b\right)}{7}.$$

Since seven is prime and 10 is not divisible by seven, $\frac{10\left(N-2b\right)}{7}$ will be a whole number only when $N-2b$ is a multiple of seven.

So we have established:

> $10N+b$ *is a multiple of seven precisely when* $N-2b$ *is.*

And what is $N-2b$? It is the number $10N+b$ with the last digit removed (giving just N) with twice the final digit ($2b$) subtracted from it.

This is the divisibility rule for seven we first described!.

Notice that $\frac{10N+b}{13}$ is a whole number precisely when

$$\frac{10N+b+39b}{13}=\frac{10(N+4b)}{13} \text{ is.}$$

And $\frac{10N+b}{17}$ is a whole number precisely when

$$\frac{10N+b-51b}{17}=\frac{10(N-5b)}{17} \text{ is.}$$

And $\frac{10N+b}{37}$ is a whole number precisely when

$$\frac{10N+b-111b}{37}=\frac{10(N-11b)}{37} \text{ is.}$$

And so on. Lots of divisibility rules for you to invent on your own now.

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Question: Do the divisibility rules constructed this way only apply to prime number divisors?

COMMENT: The argument presented here relies on the following key property of prime numbers: *If a product* $M \times N$ *is a multiple of a prime, and the first number* M *isn't, then it must be the case that the second number* N *is a multiple of that prime.* Many students and educators take this belief about primes as "obvious." Mathematicians don't! Euclid (ca. 300 BCE) was the first to prove this property of prime numbers true.

Question: Invent yet another divisibility rule for the number 11, one that involves deleting the last digit from the number.

Question: *A number is divisible by* 9 *if deleting the final digit and adding it to what remains results in an answer that is divisible by nine.* Explain!

4. Notice interesting factors of 99, 999, 9999, …, and of 101, 1001, 10001, ….

Since 111 is a multiple of 37, so is 999 and we thus observe that 1000 is one more than a multiple of 37.

Consider a number with four or more digits with leading digit b. Such a number can be written in the form:

$$b \times power\ of\ ten \times 1000 + N$$

where N is that number with the first digit removed. For example, 654274 is $6 \times 10^2 \times 1000 + 54274$.

Since $b \times power\ of\ ten \times 999$ is a multiple of 37, subtracting it from our original number won't affect our question of divisibility by 37. Thus

$$b \times power\ of\ ten \times 1000 + N$$

is a multiple of 37 precisely when

$$b \times power\ of\ ten \times 1000 + N \\ - b \times power\ of\ ten \times 999$$

is. That is, when

$$N + b \times power\ of\ ten$$

is a multiple of 37.

And what is $N + b \times power\ of\ ten$? It is the original number with the first digit removed (N) plus the digit b added to it three places further in from the left.

For example,

$$654274 = 6 \times 10^2 \times 1000 + 54274$$

is a multiple of 37 if

$$6 \times 10^2 + 54274 = 54\underline{8}74$$

is.

(Are the remainders the same?)

This explains the divisibility rule for 37 we presented.

Challenge: *To test if a number is divisible by* 11*, delete the first digit and add it two places in from the left to what remains. Check if the result is a multiple of* 11*.*

For example, we see that 241801 is not divisible by 11.

$$241801 \longrightarrow \begin{array}{r} 41801 \\ +\ 2 \\ \hline 43801 \end{array} \longrightarrow \begin{array}{r} 3801 \\ +\ 4 \\ \hline 4201 \end{array} \longrightarrow \begin{array}{r} 201 \\ +\ 4 \\ \hline 241 \end{array} \longrightarrow \begin{array}{r} 41 \\ +\ 2 \\ \hline 43 \end{array}$$

Not a multiple of 11

Why does this method work?

Since $9999 = 9 \times 11 \times 101$ and $99999 = 9 \times 41 \times 271$, we can use the same technique for creating divisibility rules for 11 (again!), 41, 101, 271, and for 99, 909, 369 and so on, by deleting the first digit of a number and subtracting it from what remains the appropriate number of places in.

Challenge: ***A number is divisible by*** 7 ***(or*** 11 ***or*** 13 ***or*** 77 ***or*** 91***) if deleting its first digit and <u>subtracting</u> it from what remains three places in from the left produces a multiple of*** 7 ***(or*** 11 ***or*** 13 ***or*** 77 ***or*** 91***, respectively!)***

Explain this rule. (HINT: What are the factors of 1001?)

Develop a divisibility test for the numbers 73 and 137 along these lines.

Question: How many different divisibility tests have we now for the number 11?

THE OPENING PUZZLER:

Division by 9 can be accomplished by via multiplication by $0.1111....$ (one ninth). Does the following example explain then what is going on with the opening puzzler?

```
12401 x 0.11111...

    1240.1
  +  124.01
  +   12.401
  +    1.2401
  +     .12401
  +     .012401
  +     .0012401
  +       ...
 ----------------
  = 1377.8888888...
```

(Recall that $0.888... = \frac{8}{9}$.)

RESEARCH CORNER:
Prove that every positive integer can be written as a sum of powers of $\frac{3}{2}$ *using the coefficients* 0*,* 1 *and* 2*. (Uniquely to boot!)*

For example, twenty in base one-and-a-half is "21202" because we have:

$$20 = \underline{\underline{2}} \times (3/2)^4 + \underline{\underline{1}} \times (3/2)^3 + \underline{\underline{2}} \times (3/2)^2 + \underline{\underline{0}} \times (3/2) + \underline{\underline{2}} \times 1$$

Now develop interesting divisibility rules for numbers written in base one-and-a-half! (For example, how do detect whether or not a number is divisible by two just be looking at its base one-and-a-half representation? Or by three? Or by seven?)

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