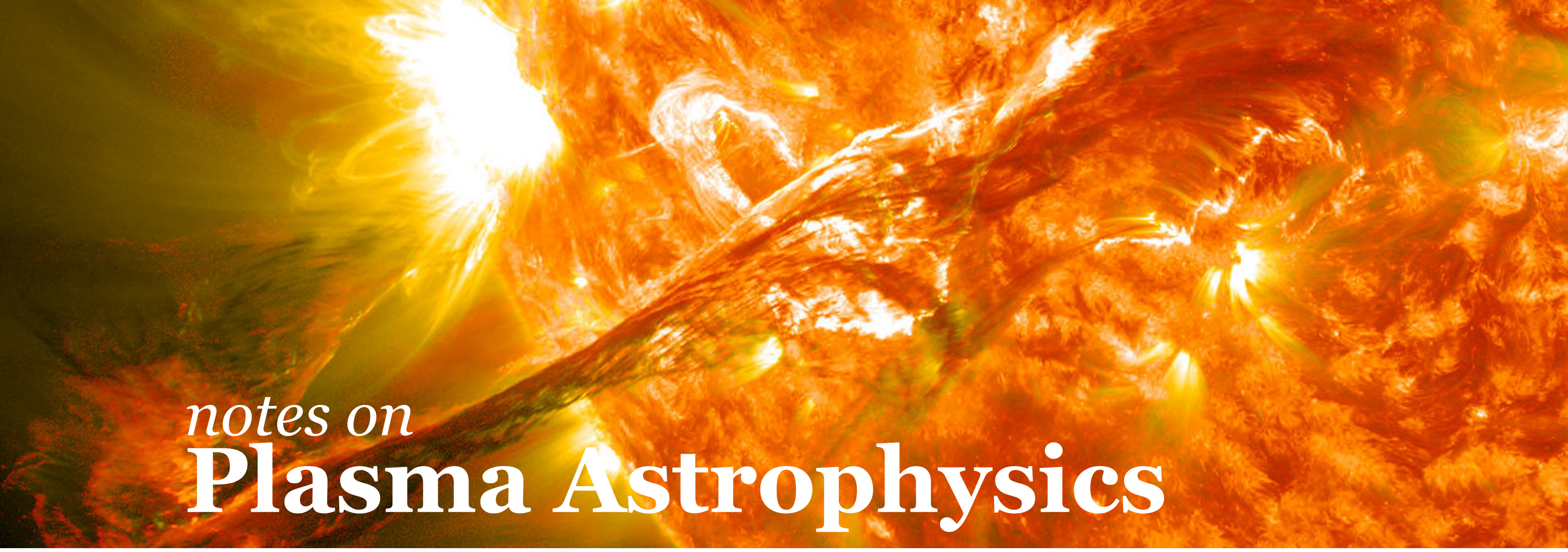
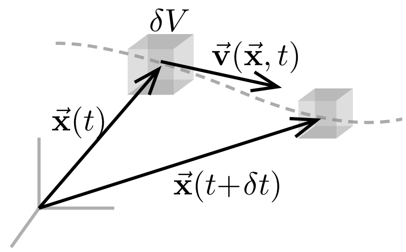


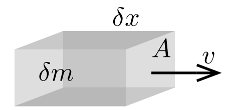
notes on
Plasma Astrophysics

Fluid dynamics



The thermodynamic state of a small volume of fluid δV is given by its density, $\rho(\vec{\mathbf{x}}, t)$ and its temperature $T(\vec{\mathbf{x}}, t)$. Let $\vec{\mathbf{v}}(\vec{\mathbf{x}}, t)$ be the velocity of the small volume located in $\vec{\mathbf{x}}$ at time t.

Time derivatives: Let $\vec{\mathbf{x}}(t+\delta t) = \vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t$. For some quantity $Q(\vec{\mathbf{x}}, t)$, the time derivative is $\dfrac{dQ}{dt} = \lim\limits_{\delta t \to 0} \dfrac{Q(\vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t, t + \delta t) - Q(\vec{\mathbf{x}}, t)}{\delta t}$. But making a Taylor expansion, $Q(\vec{\mathbf{x}} + \vec{\mathbf{v}}\delta t, t + \delta t) = Q(\vec{\mathbf{x}}, t) + \delta t \dfrac{\partial Q}{\partial t} + \delta t \vec{\mathbf{v}} \cdot \vec{\nabla} Q$, which gives us $\dfrac{dQ}{dt} = \dfrac{\partial Q}{\partial t} + \vec{\mathbf{v}} \cdot \vec{\nabla} Q$. The total derivative is called the *Lagrangian derivative* and the partial one, the *Eulerian derivative*.



Continuity equation: for the change of mass, $\dfrac{\delta m}{\delta t} = \dfrac{\rho \delta V}{\delta t} = \dfrac{\rho A \delta x}{\delta t} = \rho A v$. In general, however, $\dfrac{\partial}{\partial t}\displaystyle\int \rho\, dV = -\oint \rho \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}}$. Using the Gauss's theorem, $\dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathbf{v}}) = 0$.

Euler equation: Newton's second law: $\rho \delta V \dfrac{d\vec{\mathbf{v}}}{dt} = \delta \vec{\mathbf{F}}_{\text{body}} + \delta \vec{\mathbf{F}}_{\text{surface}}$. But $\delta \vec{\mathbf{F}}_{\text{body}} = \rho \delta V \vec{\mathbf{F}}$ (where $\mathscr{D}[\vec{\mathbf{F}}] = F/M$) and $\delta \vec{\mathbf{F}}_{\text{surface}} = -\displaystyle\oint P\, d\vec{\mathbf{A}} = -\int \vec{\nabla} P\, dV$. Inserting everything and changing the Lagrangian derivative, $\dfrac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = -\dfrac{1}{\rho}\vec{\nabla} P + \vec{\mathbf{F}}$.

Hydrodynamical perturbations

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathbf{v}}) = 0 & \text{(a)} \\
\dfrac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla})\vec{\mathbf{v}} = -\dfrac{1}{\rho}\vec{\nabla} P + \vec{\mathbf{F}} & \text{(b)} \\
\vec{\nabla}^2 \Phi = 4\pi G \rho & \text{(c)} \\
\dfrac{d}{dt}\left(\dfrac{P}{\rho^\gamma}\right) = 0 & \text{(d)}
\end{cases}
$$

Basic equations: The equations of hydrodynamics are: (a) the continuity equation, (b) the Euler equation. In an astrophysical plasma, $\vec{\mathbf{F}}$ (force per unit mass) is the gravitational field, $\vec{\mathbf{F}} = -\vec{\nabla}\Phi$, that satisfies the Poisson equation $\vec{\nabla}^2 \Phi = 4\pi G \rho$ (c).

Thermodynamics: small perturbations of pressure and density imply changes in temperature $\Longrightarrow$ it's not an isothermal process. However, small perturbations don't cause significant heat exchange with surroundings $\therefore$ adiabatic process with equation $PV^\gamma = \text{const} \implies d/dt(P/\rho^\gamma) = 0$ (d). Also, the equation of state of an ideal gas holds ($PV = Nk_BT$).

Perturbations: the plasma is in an initial equilibrium state $\{\vec{\mathbf{v}}_0 = 0,\ \rho_0,\ P_0,\ \Phi_0\}$ (at the beginning there is no displacement and then, no velocity). The perturbed values are then $\{\vec{\mathbf{v}} = \vec{\mathbf{v}}_1,\ \rho = \rho_0 + \rho_1,\ P = P_0 + P_1,\ \Phi = \Phi_0 + \Phi_1\}$ where $\rho_1 \ll \rho_0$, $P_1 \ll P_0$, $|\Phi_1| \ll |\Phi_0|$.

0th-order results:

(a) $\implies \partial_t \rho_0 = 0 \implies \rho = \text{const.}$ (a.0) (b) $\implies \vec{\nabla} P_0 = -\rho_0 \vec{\nabla} \Phi_0.$ (b.0)
(c) $\nabla^2 \Phi_0 = 4\pi G \rho_0.$ (c.0) (d) $\implies P_0/\rho_0^\gamma = \text{const.}$ (d.0)
Note that a self-gravitating, uniform, infinite gas cannot exist, since if P_0 is constant everywhere, (b.0) $\implies \Phi_0$: const, but then, (c.0) $\implies \rho_0 = 0$.

1st-order results:

(d) $\implies \dfrac{d}{dt}\left[\dfrac{P_0 + P_1}{(\rho_0+\rho_1)^\gamma}\right] = 0$, and with (d.0), $\dfrac{P_0+P_1}{(\rho_0+\rho_1)^\gamma} = \dfrac{P_0}{\rho_0^\gamma} \implies 1 + \dfrac{P_1}{P_0} = \left(1 + \dfrac{\rho_1}{\rho_0}\right)^\gamma$. Since $\rho_1/\rho_0 \ll 1$, we can expand the rhs, so that $1 + \dfrac{P_1}{P_0} \approx 1 + \gamma\dfrac{\rho_1}{\rho_0} \implies P_1 = \left(\dfrac{\gamma P_0}{\rho_0}\right)\rho_1 := c_s^2\, \rho_1$ **(d.1)**.

(a), (a.0) $\implies \partial_t \rho_1 + \vec{\nabla}[(\rho_0+\rho_1)\vec{\mathbf{v}}_1] = 0$. The term $\rho_1 \vec{\mathbf{v}}_1$ is smaller than first order, so it goes to zero, and ρ_0 is constant. This implies $\partial_t \rho_1 + \rho_0 \vec{\nabla} \cdot \vec{\mathbf{v}}_1 = 0$ **(a.1)**.

(b) $\implies (\rho_0+\rho_1)\Big[\partial_t \vec{\mathbf{v}}_1 + (\vec{\mathbf{v}}_1 \cdot \vec{\nabla})\vec{\mathbf{v}}_1\Big] = -\vec{\nabla}(\underline{P_0} + P_1) - \vec{\nabla}(\underline{\Phi_0} + \Phi_1)(\underline{\rho_0} + \rho_1)$, but the underlined terms cancel each other due to (b.0). The second term of the Lagrangian derivative is small, and so are the other terms involving ρ_1. Then, we have $\rho_0 \partial_t \vec{\mathbf{v}}_1 = -\vec{\nabla} P_1 - \rho_0 \vec{\nabla}\Phi_1$. Using (d.1), this becomes $\rho_0 \partial_t \vec{\mathbf{v}}_1 = -c_s^2 \vec{\nabla}\rho_1 - \rho_0 \vec{\nabla}\Phi_1$ **(b.1)**.

(c), (c.0) $\implies \vec{\nabla}^2 \Phi_1 = 4\pi G \rho_1$ **(c.1)**.

Sound equation: if gravity is negligible, for example, in the atmosphere, we have, for eq. (b.1), and taking the divergence in both sides, $\vec{\nabla} \cdot \Big[\rho_0 \partial_t \vec{\mathbf{v}}_1 = -c_s^2 \vec{\nabla}\rho_1\Big]$. Rearranging the lhs, we can have $\partial_t(\rho_0 \vec{\nabla}\cdot\vec{\mathbf{v}}_1)$, that we can substitute using (a.1), so that we have $\dfrac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \vec{\nabla}^2 \rho_1$. This is a wave equation, and so we know that c_s has to be the speed of sound. This means that perturbations travel with the speed of sound (that was evaluated in the 0th order).

Periodic perturbations: we take the equations (a.1), (b.1) and (c.1). If the small perturbations are periodic, we can write solutions for each variable h of the form $h_{\text{max}} e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{x}} - \omega t)}$, which is the same as taking the Fourier transform of each equation; moreover, $\mathcal{F}\{\partial_t\} = -i\omega$, $\mathcal{F}\{\vec{\nabla}\} = i\vec{\mathbf{k}}$. Then, the system of equations becomes: $\omega\rho_1 = \rho_0 \vec{\mathbf{k}} \cdot \vec{\mathbf{v}}_1$ **(a1F)** $\omega\rho_0 \vec{\mathbf{v}}_1 = \vec{\mathbf{k}} c_s^2 \rho_1 + \vec{\mathbf{k}}\rho_0 \Phi_1$ **(b1F)** $\vec{\mathbf{k}}^2 \Phi_1 = -4\pi G \rho_1$ **(c1F)**

Jeans instability: (a1F), ($\mathbf{v}_1$ from b1F) $\implies \omega^2 \rho_1 = \vec{\mathbf{k}} \cdot \vec{\mathbf{k}} c_s^2 \rho_1 + \vec{\mathbf{k}} \cdot \vec{\mathbf{k}} \rho_0 \Phi_1$. (c1F), ($\cancel{\rho_1}$) $\implies \omega^2 = k^2 c_s^2 - 4\pi G \rho_0$. We write this dispersion relation as $\omega^2 = c_s^2 \left[k^2 - \dfrac{4\pi G\rho_0}{c_s^2}\right] := c_s^2(k^2 - k_J^2)$. Notice that ω is real only if $k_J < k$ (oscillation), but if $k_J > k$, ω becomes complex and the exponential on the temporal part of each variable becomes real, leading to an exponential grow. This means that if the size of the perturbation is larger than $\lambda_J = 2\pi/k_J$ and self-gravity overpowers acoustic waves. The mass that corresponds to λ_J is called the *Jeans mass*, and is calculated as $M = \dfrac{4}{3}\pi\lambda_J^3 \rho_0$. Using (d.1) and the equation of state of the ideal gas, as well as the approximation $\gamma \approx 1$ (slow perturbations are approx. isothermal), then, $M_J = \dfrac{4}{3}\pi^{5/2}\rho_0^{-1/2}\left(\dfrac{k_B T}{G m}\right)^{3/2}$ (m is the mass of a molecule).

Orthogonal curvilinear coordinates

Transformations

Cylindrical

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

Spherical

$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

Line elements

Cartesian

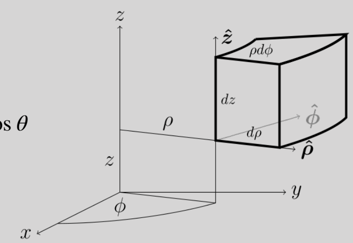
$ds^1 = dx \quad ds^2 = dy \quad ds^3 = dz$

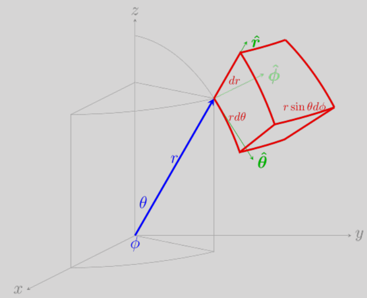
Cylindrical

$ds^1 = dr \quad ds^2 = r\,d\phi \quad ds^3 = dz$

Spherical

$ds^1 = dr \quad ds^2 = r\,d\theta \quad ds^3 = r\sin\theta\,d\phi$





Scale factors: $ds^i = h_i dq^i$

Unit vectors: from the transformation, $\hat{\mathbf{e}}_i := \dfrac{\partial\vec{\mathbf{r}}}{\partial s^i} = \dfrac{1}{h_i}\dfrac{\partial\vec{\mathbf{r}}}{\partial q^i}$. Example, sph.: $\hat{\phi} = \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}})$
$= \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}(r\sin\theta\cos\phi\,\hat{\mathbf{x}} + r\sin\theta\sin\phi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}) = -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}$.

Transformation of vectors (contravariant) vs covectors (covariant): for a differential, $dx^i = \dfrac{\partial x^i}{\partial q^j}dq^j$ (ex.: velocity, contravariant), and for a derivative $\dfrac{\partial\xi}{\partial x^i} = \dfrac{\partial q^j}{\partial x^i}\dfrac{\partial\xi}{\partial q^j}$ (gradient, covariant). (See Notes on General Relativity).

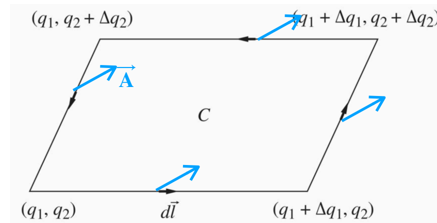
Gradient: expanding $f(\vec{\mathbf{r}} + \Delta\vec{\mathbf{r}}) = f(x+\Delta x, y+\Delta y, z+\Delta z) \approx f(x,y,z) + \Delta\vec{\mathbf{r}}\cdot\vec{\nabla}f$. The second term expresses that the maximum change of the function occurs when the displacement is in the direction of the gradient. In a general direction $\hat{\mathbf{n}}$, we define the directional derivative: $df/dn := \hat{\mathbf{n}}\cdot\vec{\nabla}f$. In an orthogonal coord. syst., $\hat{\mathbf{n}}$ is one of the unit vectors $\hat{\mathbf{e}}_i$, so that the total gradient is
$\vec{\nabla}f = \sum_i \hat{\mathbf{e}}_i\dfrac{\partial f}{\partial s_i} = \sum_i \hat{\mathbf{e}}_i\dfrac{1}{h_i}\dfrac{\partial f}{\partial q_i}$.

Divergence: thinking of a vector field as the flow of the field quantity, the divergence is the normalized total flow out of the infinitesimal volume elements centered at each point: $\vec{\nabla}\cdot\vec{\mathbf{A}} := \lim_{\Delta V\to 0}\dfrac{1}{\Delta V}\oint_S \vec{\mathbf{A}}\cdot d\vec{\mathbf{S}}$ (div. theorem), where $d\vec{\mathbf{S}}_3 = ds_1 ds_2\hat{\mathbf{e}}_3 = h_1h_2dq_1\,dq_2\hat{\mathbf{e}}_3$ and cyclic permutations. In a volume element ΔV in which the coordinates of the faces are q_i, $q_i + \Delta q_i$ in each direction (showing only a pair of faces),

$$\vec{\nabla}\cdot\vec{\mathbf{A}} = \lim_{V\to 0}\frac{1}{h_1h_2h_3\Delta q_1\Delta q_2\Delta q_3}\left[\Delta q_2\Delta q_3\left(A_1h_2h_3\Big|_{q_1+\Delta q_1,q_2,q_3} - A_1h_2h_3\Big|_{q_1,q_2,q_3}\right) + \text{cycl. perm.}\right]$$
$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(A_1h_2h_3) + \text{cycl. perm.}\right]$$

Laplacian: combining divergence and gradient, $\nabla^2 f = \vec{\nabla}\cdot\vec{\nabla}f = \dfrac{1}{h_1h_2h_3}\left(\dfrac{\partial}{\partial q_1}\left[\dfrac{h_2h_3}{h_1}\dfrac{\partial f}{\partial q_1}\right] + \text{cycl. perm.}\right)$

Curl: evaluates the circulation of a vector field at each point (how non conservative it is), with the direction determined by the right hand rule. So, in a curve, $\vec{\nabla} \times \vec{\mathbf{A}} = \lim_{S \to 0} \frac{1}{S} \oint \vec{\mathbf{A}} \cdot d\vec{\ell}$. For example, let us take the loop in the $q_1 q_2$ plane (dir.: q_3):



$$(\vec{\nabla} \times \vec{\mathbf{A}})_3 = \frac{1}{h_1 h_2 \Delta q_1 \Delta q_2} \Big[-A_1 \Delta s_1 \Big|_{(q_1 + \Delta q_1/2,\, q_2 + \Delta q_2)} + A_1 \Delta s_1 \Big|_{(q_1 + \Delta q_1/2,\, q_2)} - A_2 \Delta s_2 \Big|_{(q_1,\, q_2 + \Delta q_2)} + A_2 \Delta s_2 \Big|_{(q_1 + \Delta q_1,\, q_2 + \Delta q_2/2)} \Big]$$

$$= \frac{1}{h_1 h_2 \Delta q_1 \Delta q_2} \left[-\frac{\partial}{\partial q_2}(A_1 h_1) + \frac{\partial}{\partial q_1}(A_2 h_2) \right] \Delta q_1 \Delta q_2 = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial q_1}(A_2 h_2) - \frac{\partial}{\partial q_2}(A_1 h_1) \right].$$

In general, we can write the determinant $\vec{\nabla} \times \vec{\mathbf{A}} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$.

Tensor derivatives

Metric tensor and scale factors: $g_{ij} = h_i h_j$. Ex.: polar coords., metric: $d\mathcal{S}^2 = dr^2 + r^2 d^2\theta \implies g_{rr} = 1,\ g_{\theta\theta} = r^2$ (all others 0); $h_r = 1,\ h_\theta = r$.

Bases. There are three standard bases in Euclidean space, in which the components can be expressed: a) contravariant A^i, b) covariant A_i, c) physical $A_{(i)}$. The difference is in the units: $A_i = g_{ij} A^j$; $A_{(i)} = h_{(i)} A^i$ (ex., for speed $v^\theta = \dot{\theta}$, $v_\theta = r^2 \dot{\theta}$ but $v_{(\theta)} = r\dot{\theta}$; the last one has units of speed).

Covariant derivative: $\frac{\partial \vec{\mathbf{A}}}{\partial x^j} = \frac{\partial}{\partial x^j}(A^i \hat{\mathbf{e}}_i) = \frac{\partial A^i}{\partial x^j} \hat{\mathbf{e}}_i + \frac{\partial \hat{\mathbf{e}}_i}{\partial x^j} A^i$ but $\frac{\partial \hat{\mathbf{e}}_i}{\partial x^j} = \Gamma^k_{ij} \hat{\mathbf{e}}_k$ (*warning!* this is for a contravariant representation of a vector, not for the physical components). Note that $\partial \vec{\mathbf{A}} / \partial x^j = A^i_{\ ;j} \hat{\mathbf{e}}_i$.

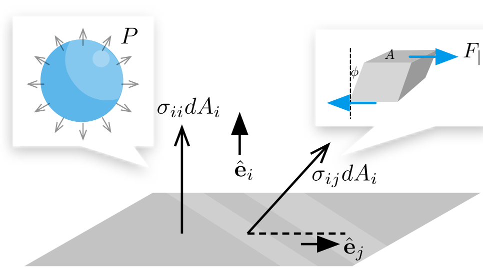
Divergence from covariant derivative: $A^i_{\ ;i} = A^i_{\ ,i} + \Gamma^i_{ij} A^j$, but we can write $\Gamma^i_{ij} = \cdots = \frac{1}{2} g^{ki} g_{ki,j}$. Now, by *Jacobi's Formula*, we get $\Gamma^i_{ij} A^j = A^j (\partial_j g)/g$, where g is the determinant of the metric. ($\rightleftarrows$ dum. index)

$\implies A^i_{\ ;i} = A^i_{\ ,i} + A^i \frac{1}{2g} \partial_i g = A^i_{\ ,i} + A^i \frac{1}{\sqrt{g}} \partial_i \sqrt{g} = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} A^i)$. Now we have $\sqrt{g} = h_{(1)} h_{(2)} h_{(3)}$ and $A^i_{\ ;i} = \frac{1}{h_{(1)} h_{(2)} h_{(3)}} \frac{\partial}{\partial x^i} \left(\frac{h_{(1)} h_{(2)} h_{(3)}}{h_{(i)}} A_{(i)} \right)$, which are the physical components of the divergence.

Divergence of a tensor: The covariant derivative is $T^{ij}_{\ \ ;i} = T^{ij}_{\ \ ,i} + \Gamma^i_{ik} T^{kj} + \Gamma^j_{ik} T^{ik}$. For the first two terms, we apply the same treatment as the divergence of a vector, and the last one has to be rearranged later in order to get the physical components.

Navier-Stokes equations

Navier-Stokes equations: $\frac{\partial \vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}} \cdot \vec{\nabla}) \vec{\mathbf{v}} = -\frac{1}{\rho} \left(\vec{\nabla} P + \vec{\nabla} \cdot \overleftrightarrow{\sigma} \right) - \vec{\nabla} \Phi$



Cauchi stress tensor $\overleftrightarrow{\sigma}$: $\mathcal{D}[\sigma_{ij}] = F/L^2$. The dot product with the tensor generates a vector, and that vector should be in their physical components. The total stress, or force per unit area, in the $\vec{\mathbf{n}}$ direction is $\vec{\mathbf{T}}^{(n)}_j = \vec{\mathbf{n}} \cdot \overleftrightarrow{\sigma}$.

Equations of magneto-hydrodynamics

Curl of a cross product: $\vec{\nabla}\times(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = (\vec{\mathbf{B}}\cdot\vec{\nabla})\vec{\mathbf{A}} - \vec{\mathbf{B}}(\vec{\nabla}\cdot\vec{\mathbf{A}}) + (\vec{\nabla}\cdot\vec{\mathbf{B}})\vec{\mathbf{A}} - (\vec{\mathbf{A}}\cdot\vec{\nabla})\vec{\mathbf{B}}$. First, we apply the product rule (marking which vector $\vec{\nabla}$ is acting on) and then we expand using the BAC-CAB rule.

Gradient of a dot product: $\vec{\nabla}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}) = \dot{\vec{\nabla}}(\dot{\vec{\mathbf{A}}}\cdot\vec{\mathbf{B}}) + \dot{\vec{\nabla}}(\vec{\mathbf{A}}\cdot\dot{\vec{\mathbf{B}}})$ (the dot marks where the derivative acts on). But applying the BAC-CAB rule to the cross product of a curl, $\vec{\mathbf{A}}\times(\vec{\nabla}\times\vec{\mathbf{B}}) = \dot{\vec{\nabla}}(\vec{\mathbf{A}}\cdot\dot{\vec{\mathbf{B}}}) - (\vec{\mathbf{A}}\cdot\vec{\nabla})\vec{\mathbf{B}}$, one of the terms appear. Making $\vec{\mathbf{A}} \leftrightarrows \vec{\mathbf{B}}$ and substituting, we find $\vec{\nabla}(\vec{\mathbf{A}}\cdot\vec{\mathbf{B}}) = (\vec{\mathbf{A}}\cdot\vec{\nabla})\vec{\mathbf{B}} + (\vec{\mathbf{B}}\cdot\vec{\nabla})\vec{\mathbf{A}} + \vec{\mathbf{A}}\times(\vec{\nabla}\times\vec{\mathbf{B}}) + \vec{\mathbf{B}}\times(\vec{\nabla}\times\vec{\mathbf{A}})$.

Ohm's law: $\vec{\mathbf{j}} = \sigma\vec{\mathbf{E}}$, but since the magnetic field also contributes to the force on charges, $\vec{\mathbf{j}} = \sigma(\vec{\mathbf{E}} + \vec{\mathbf{v}}\times\vec{\mathbf{B}})$.

Ampère-Maxwell equation: for astrophysical plasmas with $v \ll c$, we ignore the displacement current term. So, $\vec{\nabla}\times\vec{\mathbf{B}} = \mu_0\vec{\mathbf{j}}$.

Electric field: combining Ampère's and Ohm's laws, $\vec{\mathbf{E}} = \dfrac{\vec{\nabla}\times\vec{\mathbf{B}}}{\mu_0\sigma} - \vec{\mathbf{v}}\times\vec{\mathbf{B}}$. For astrophysical plasmas, $\vec{\mathbf{B}}$ is more relevant than $\vec{\mathbf{E}}$, because charges are well mixed.

Induction equation: Faraday's law: $\dfrac{\partial\vec{\mathbf{B}}}{\partial t} = -\vec{\nabla}\times\vec{\mathbf{E}}$. Substituting the electric field, $\dfrac{\partial\vec{\mathbf{B}}}{\partial t} = \vec{\nabla}\times(\vec{\mathbf{v}}\times\vec{\mathbf{B}}) + \eta\nabla^2\vec{\mathbf{B}}$, where $\eta = 1/(\mu_0\sigma)$. [Application of the "BAC-CAB" rule for gradients, one of the terms will be $\vec{\nabla}\cdot\vec{\mathbf{B}} = 0$. $\vec{\nabla}\times(\vec{\mathbf{v}}\times\vec{\mathbf{B}})$ is called the *advection* term.]

Advection vs convection: convection is the movement of a fluid mainly due to density gradients created by thermal gradients; advection is the more general transport of material or physical quantity by the velocity of the fluid.

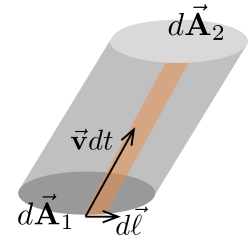
Advection equation: the general equation $\dfrac{\partial\psi}{\partial t} + \vec{\nabla}\cdot(\psi\vec{\mathbf{v}}) = 0$ states the advection for a conserved quantity described by a scalar field ψ due to the transport in the velocity field $\vec{\mathbf{v}}$. If $\vec{\nabla}\cdot\vec{\mathbf{v}} = 0$ (incompressible flow / solenoidal) $\implies \dfrac{\partial\psi}{\partial t} + \vec{\mathbf{v}}\cdot\vec{\nabla}\psi = 0$.

Euler equation: modification: $\vec{\mathbf{F}} \to \vec{\mathbf{F}} + \dfrac{1}{\rho}\vec{\mathbf{j}}\times\vec{\mathbf{B}}$. Substituting $\vec{\mathbf{j}}$ (with Ampère's law), using the gradient of a dot product when both vectors are equal, $\dfrac{\partial\vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}}\cdot\vec{\nabla})\vec{\mathbf{v}} = \vec{\mathbf{F}} - \dfrac{1}{\rho}\vec{\nabla}\left(P + \dfrac{B^2}{2\mu_0}\right) + \dfrac{\vec{\mathbf{B}}\cdot\vec{\nabla}}{\mu_0\rho}\vec{\mathbf{B}}$. The first new term is the magnetic field pressure and the second term, the magnetic tension force, which tends to straighten magnetic field lines.

Summary: for an astrophysical plasma, there is a system of two vector equations:

$$\begin{cases} \frac{\partial\vec{\mathbf{B}}}{\partial t} = \vec{\nabla}\times(\vec{\mathbf{v}}\times\vec{\mathbf{B}}) + \eta\nabla^2\vec{\mathbf{B}} \\ \frac{\partial\vec{\mathbf{v}}}{\partial t} + (\vec{\mathbf{v}}\cdot\vec{\nabla})\vec{\mathbf{v}} = \vec{\mathbf{F}} - \frac{1}{\rho}\vec{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{\vec{\mathbf{B}}\cdot\vec{\nabla}}{\mu_0\rho}\vec{\mathbf{B}} \end{cases}$$

The usual goal is to solve for $\vec{\mathbf{v}}$ and $\vec{\mathbf{B}}$. MHD is *ideal* if $\eta \to 0$. This equation has SI units, for Gaussian, we use the transformation $\dfrac{\vec{\mathbf{B}}_{[G]}}{\vec{\mathbf{B}}_{[SI]}} = \sqrt{\dfrac{4\pi}{\mu_0}}$, which leaves the first equation invariant and the second one, with $\mu_0 \to 4\pi$.



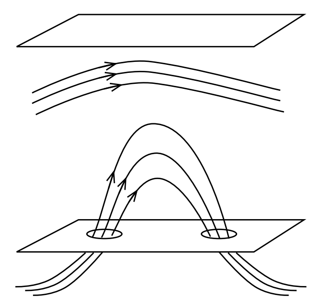
Flux freezing

Magnetic Reynolds number: Comparison of the two terms of the induction equation: $\mathscr{R}_M \approx \dfrac{vB/L}{\eta B/L^2} \approx \dfrac{vL}{\eta}$. For an astrophysical plasma, L is very large, compared to a laboratory plasma. Then, the induction equation can be written as $\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \eta \nabla^2 \vec{\mathbf{B}}$ for a laboratory plasma and $\dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}})$ for an astrophysical plasma.

Alfvén's theorem of flux freezing: The Lagrangian derivative of the magnetic flux is $\dfrac{d\Phi}{dt} = \displaystyle\int_S \dfrac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}} + \int \vec{\mathbf{B}} \cdot \dfrac{d}{dt} d\vec{\mathbf{A}}$. But the extra area swept by the motion of $d\vec{\mathbf{A}}$ is the area of the side of the cylinder, $dt \displaystyle\oint_C \vec{\mathbf{v}} \times d\vec{\ell}$. Then, $\dfrac{d\Phi}{dt} = \displaystyle\int_S \dfrac{\partial \vec{\mathbf{B}}}{\partial t} \cdot d\vec{\mathbf{A}} + \oint_C \vec{\mathbf{B}} \times \vec{\mathbf{v}} \cdot d\vec{\ell}$

$$\implies \frac{d\Phi}{dt} = \int_S \left[\frac{\partial \vec{\mathbf{B}}}{\partial t} - \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) \right] \cdot d\vec{\mathbf{A}} = 0.$$

Neutron stars: $B_1 A_1 = B_2 A_2 \implies B_2 = B_1 R_1^2 / R_2^2$. If a star like the Sun collapsed to the radius of a neutron star, the magnetic field would become $B_2 \sim 10^7$ T, the order of magnitude of typical stars, but insufficient to explain the one of magnetars.



Magnetic buoyancy

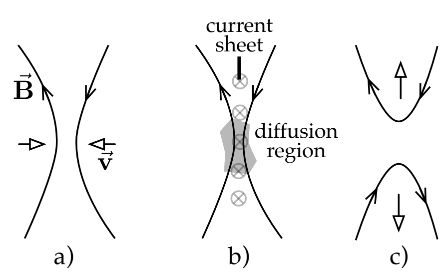
Consider a horizontal cylindrical region of concentrated magnetic field, called *magnetic flux tube*. The flux is frozen, which means that both the plasma and the magnetic field respond immediately to changes. If there is hydrostatic balance, $P_{\text{ext}} = P_{\text{int}} + \dfrac{B^2}{2\mu_0}$. Then, $P_{\text{int}} < P_{\text{ext}}$. But if the gas is ideal, $P \propto \rho T$, and if the temperatures are the same, $\rho_{\text{int}} < \rho_{\text{ext}}$, making the whole section of the tube buoyant.

Magnetic reconnection

a) Opposing magnetic field lines approach each other.

b) When the lines are too close, $\eta \nabla^2 \vec{\mathbf{B}}$ can no longer be ignored and a diffusion region is created. The magnetic field configuration requires the existence of a current sheet.

c) Magnetic field lines are reconfigured. The process makes a conversion of magnetic energy into kinetic energy. Plasma is propelled. The process is sustainable: lowering the magnetic field in the center also lowers the pressure, causing plasma to get "sucked in" and the process to continue.



Linearization of MHD

For every variable h of the ideal MHD equations, we expand it such as $h = h_0 + \varepsilon h_1$ with $\varepsilon \ll 1$. For the equilibrium (0th order), the velocity vanishes, so $\vec{\mathbf{v}} = \varepsilon \vec{\mathbf{v}}_1$.

Ideal MHD equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{\mathbf{v}}) = 0 & (a) \\ \rho \dfrac{\partial \vec{\mathbf{v}}}{\partial t} + \rho (\vec{\mathbf{v}} \cdot \vec{\nabla}) \vec{\mathbf{v}} = -\vec{\nabla} P + \dfrac{1}{\mu_0} (\vec{\nabla} \times \vec{\mathbf{B}}) \times \vec{\mathbf{B}} + \vec{\mathbf{f}} & (b) \\ \dfrac{d}{dt} (P \rho^{-\gamma}) = 0 \ \leftarrow \text{equation of state} & (c) \\ \dfrac{\partial \vec{\mathbf{B}}}{\partial t} = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}) & (d) \end{cases}$$

0th-order:

(a),(d) $\Longrightarrow \partial_t \rho_0 = \partial_t B_0 = 0$ (a.0, d.0)
(c) $\Longrightarrow P_0 \rho_0^{-\gamma} = \text{const}$ (c.0)
(b) $\Longrightarrow 0 = -\vec{\nabla} P_0 + (\vec{\nabla} \times \vec{\mathbf{B}}_0) \times \vec{\mathbf{B}}_0 / \mu_0 + \vec{\mathbf{f}}_0$ (b.0)

First order:

Lagr. der.: $\frac{d}{dt}(\cancel{Q_0} + Q_1) = \frac{\partial}{\partial t}(\cancel{Q_0} + Q_1) + \vec{\mathbf{v}} \cdot \vec{\nabla}(Q_0 + \cancel{Q_1})$,
since Q_0 is in equilibrium and terms of order vQ_1 are small.
(a) $\Longrightarrow \partial_t \rho_1 + \vec{\nabla} \cdot (\rho_0 \vec{\mathbf{v}}) = 0 \Longrightarrow \partial_t \rho_1 + \rho_0 (\vec{\nabla} \cdot \vec{\mathbf{v}}) + (\vec{\mathbf{v}} \cdot \vec{\nabla}) \rho_0 = 0$ (a.1)
(b) $\Longrightarrow \rho_0 \partial_t \vec{\mathbf{v}} = -\vec{\nabla} P_1 + [(\vec{\nabla} \times \vec{\mathbf{B}}_0) \times \vec{\mathbf{B}}_1 + (\vec{\nabla} \times \vec{\mathbf{B}}_1) \times \vec{\mathbf{B}}_0] / \mu_0 + \vec{\mathbf{f}}_1$ (b.1) (one uses (b.0) to cancel terms, and terms of the order $\rho_1 v, B_1^2$ are too small)
(c) $\Longrightarrow \rho_0^{-\gamma} \frac{d}{dt} P_1 - \gamma P_0 \rho_0^{-\gamma-1} \frac{d}{dt} \rho_1 = 0 \Longrightarrow \frac{d}{dt} P_1 = c_s^2 \frac{d}{dt} \rho_1$ (c.1) ($c^2 = \gamma P_0 / \rho_0$)
(d) $\Longrightarrow \partial_t \vec{\mathbf{B}}_1 = \vec{\nabla} \times (\vec{\mathbf{v}} \times \vec{\mathbf{B}}_0)$ (d.1)
(c.1) $\Longrightarrow \frac{\partial}{\partial t} P_1 + \vec{\mathbf{v}} \cdot \vec{\nabla} P_0 = c_s^2 \left[\frac{\partial}{\partial t} \rho_1 + \vec{\mathbf{v}} \cdot \vec{\nabla} \rho_0 \right]$; ($\rightarrow$a.1) $\Longrightarrow \frac{\partial}{\partial t} P_1 + (\vec{\mathbf{v}} \cdot \vec{\nabla}) P_0 + c_s^2 \rho_0 (\vec{\nabla} \cdot \vec{\mathbf{v}}) = 0$ (e.1)

Lagrangian displacement: $\vec{\xi}$ is defined as $\vec{\mathbf{u}} = \frac{d}{dt} \vec{\xi} \approx \partial_t \vec{\xi}$.

This allows us to integrate (a.1) in time: $\rho_1 + \vec{\nabla} \cdot (\rho_0 \vec{\xi}) = \text{const}$ (a.2). If at $t = 0$ every perturbation is chosen as 0 except $\xi(\vec{\mathbf{r}}_0, 0)$; then the constant is zero. We can repeat the procedure with the other variables, P_1 and $\vec{\mathbf{B}}_1$ (e.2), (d.2).

(b.1) becomes $\rho_0 \partial_t^2 \vec{\xi} = \vec{\mathbf{F}}_V(\vec{\xi})$, with $\vec{\mathbf{F}}_V$ the force per unit volume,
$\vec{\mathbf{F}}_V(\vec{\xi}) = -\vec{\nabla} P_1 + [(\vec{\nabla} \times \vec{\mathbf{B}}_0) \times \vec{\mathbf{B}}_1 + (\vec{\nabla} \times \vec{\mathbf{B}}_1) \times \vec{\mathbf{B}}_0] / \mu_0 + \vec{\mathbf{f}}_1$ (b.2)
$\vec{\nabla} \times \vec{\mathbf{B}}_1 = \vec{\nabla} \times \vec{\nabla} \times (\vec{\xi} \times \vec{\mathbf{B}}_0)$.

Linearized ideal MHD

$$\begin{cases} \rho_1 = -(\vec{\xi} \cdot \vec{\nabla}) \rho_0 - \rho_0 (\vec{\nabla} \cdot \vec{\xi}) & (a.2) \\ P_1 = -(\vec{\xi} \cdot \vec{\nabla}) P_0 - \rho_0 c_s^2 (\vec{\nabla} \cdot \vec{\xi}) & (e.2) \\ \vec{\mathbf{B}}_1 = \vec{\nabla} \times (\vec{\xi} \times \vec{\mathbf{B}}_0) & (d.2) \\ \rho_0 \partial_t^2 \vec{\xi} = -\vec{\nabla} P_1 + [(\vec{\nabla} \times \vec{\mathbf{B}}_0) \times \vec{\mathbf{B}}_1 + \\ \qquad (\vec{\nabla} \times \vec{\mathbf{B}}_1) \times \vec{\mathbf{B}}_0] / \mu_0 + \vec{\mathbf{f}}_1 & (b.2) \end{cases}$$

Waves and oscillations

Plasma oscillations: if the electrons in a plasma are displaced from a uniform background of ions, the ions exert a restoring force; the resulting oscillations have the *plasma frequency* $\omega_p = \sqrt{n_0 e^2 / (\epsilon_0 m)}$ (see Chen § 4.3). This is an electrostatic oscillation (astrophysical plasmas are dominated by magnetic fields).

Hydromagnetic waves: low-frequency ion oscillations in presence of a magnetic field.

Alfvén waves: taking the Fourier transforms ($\mathscr{F}\{\partial_t\} = -i\omega$, $\mathscr{F}\{\vec{\nabla}\} = i\vec{\mathbf{k}}$) for the linearized versions of the ideal MHD equations, we get the equations on the right (tildes omitted). In those equations we also assumed no external force and a uniform unperturbed state.

$$\begin{cases} \rho_1 = -i \rho_0 (\vec{\mathbf{k}} \cdot \vec{\xi}) & (a2F) \\ P_1 = -\rho_0 c_s^2 (\vec{\mathbf{k}} \cdot \vec{\xi}) & (e2F) \\ \vec{\mathbf{B}}_1 = i \vec{\mathbf{k}} \times (\vec{\xi} \times \vec{\mathbf{B}}_0) & (d2F) \\ -\omega^2 \rho_0 \vec{\xi} = -i \vec{\mathbf{k}} P_1 + i [(\vec{\nabla} \times \vec{\mathbf{B}}_1) \times \vec{\mathbf{B}}_0] / \mu_0 & (b2F) \end{cases}$$

Substituting $\rho_1, P_1, \vec{\mathbf{B}}_1$ in (b2F), we get $\omega^2 \rho_0 \vec{\xi} = \rho_0 c_s^2 \vec{\mathbf{k}} (\vec{\mathbf{k}} \cdot \vec{\xi}) + \{\vec{\mathbf{k}} \times [\vec{\mathbf{k}} \times (\vec{\xi} \times \vec{\mathbf{B}}_0)] \times \vec{\mathbf{B}}_0\} / \mu_0$ (f).

Let $\vec{\mathbf{B}}_0 = B_0 \vec{\mathbf{e}}_b$, and $c_a^2 := B_0^2 / (\mu_0 \rho_0)$ (the Alfvén speed) and we get for (f):
$\omega^2 \vec{\xi} = c_s^2 \vec{\mathbf{k}} (\vec{\mathbf{k}} \cdot \vec{\xi}) + c_a^2 [\vec{\mathbf{k}} \times [\vec{\mathbf{k}} \times (\vec{\xi} \times \hat{\mathbf{e}}_b)] \times \hat{\mathbf{e}}_b]$.

For the case in which the plasma compressibility can be neglected, $c_s \ll c_a$, and using the BAC-CAB rule twice, we have for (f): $\omega^2 \vec{\xi} = c_a^2 \left[(\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)^2 \vec{\xi} + \left(\vec{\mathbf{k}} \cdot \vec{\xi} - [\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b][\vec{\xi} \cdot \hat{\mathbf{e}}_b] \right) \vec{\mathbf{k}} + (\vec{\mathbf{k}} \cdot \vec{\xi})(\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)\hat{\mathbf{e}}_b \right]$ (f')

Making (f') $\cdot\, \hat{\mathbf{e}}_b$, all the terms in the rhs cancel and then, $\omega^2 \vec{\xi} \cdot \hat{\mathbf{e}}_b = 0 \implies \vec{\xi} \cdot \hat{\mathbf{e}}_b = 0$ if we want $\omega^2 \neq 0$. Making (f') $\cdot\, \vec{\mathbf{k}}$, we get $\omega^2 \vec{\xi} \cdot \vec{\mathbf{k}} = c_a^2 (\vec{\mathbf{k}} \cdot \vec{\xi}) k^2 - c_a^2 (\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)(\vec{\xi} \cdot \hat{\mathbf{e}}_b) k^2$, and using the simplification just discovered, we get $(\omega^2 - k^2 c_a^2)(\vec{\mathbf{k}} \cdot \vec{\xi}) = 0$. This is a wave equation (in transformed space), and its solutions are $\vec{\mathbf{k}} \cdot \vec{\xi} = 0$ (Alfvén waves) or $(\omega^2 - k^2 c_a^2) = 0$ (Compressible Alfvén waves).

If we introduce $\vec{\mathbf{k}} \cdot \vec{\xi} = 0$, $\vec{\xi} \cdot \hat{\mathbf{e}}_b = 0$ into (f'), we get $\omega^2 \vec{\xi} = c_a^2 (\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)^2 \vec{\xi}$. The dispersion relation for the Alfvén waves is, then, $\omega^2 = k^2 c_a^2 \cos^2\theta$, where θ is the angle between $\vec{\mathbf{B}}_0$ and $\vec{\mathbf{k}}$.

Features of Alfvén waves: Alfvén waves are transverse waves, with incompressible *perturbations* (not plasma) and group speed equal to the Alfvén speed. The magnetic field perturbations are perpendicular to the magnetic field $\vec{\mathbf{B}}_0$ and the solution is general, not only applies to a first-order approximation.

Magnetosonic waves: taking (f) without the condition $c_s \ll c_a$, and expanding the triple vector product, we get $\omega^2 \vec{\xi} = c_a^2 \left[(\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)^2 \vec{\xi} + \left(\vec{\mathbf{k}} \cdot \vec{\xi} - [\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b][\vec{\xi} \cdot \hat{\mathbf{e}}_b] \right) \vec{\mathbf{k}} + (\vec{\mathbf{k}} \cdot \vec{\xi})(\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)\hat{\mathbf{e}}_b \right] + c_s^2 (\vec{\mathbf{k}} \cdot \vec{\xi}) \vec{\mathbf{k}}$ (g) (which is f' + the term with c_s^2).

(g) $\cdot\, \hat{\mathbf{e}}_b \implies \omega^2 \vec{\xi} \cdot \hat{\mathbf{e}}_b = c_s^2 (\vec{\mathbf{k}} \cdot \vec{\xi})(\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)$ (g.1)

(g) $\cdot\, \vec{\mathbf{k}} \implies [\omega^2 - k^2(c_s^2 + c_a^2)](\vec{\mathbf{k}} \cdot \vec{\xi}) = -k^2 c_a^2 (\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)(\vec{\xi} \cdot \hat{\mathbf{e}}_b)$ (g.2)

If $(\vec{\mathbf{k}} \cdot \vec{\xi}) = 0 \implies (\xi \cdot \hat{\mathbf{e}}_b) = 0 \implies$ Alfvén waves. If $(\vec{\mathbf{k}} \cdot \vec{\xi}) \neq 0$, we can do (g.2) $(\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)$ and use (g.1) to obtain $[\omega^2 - k^2 c_s^2 - k^2 c_a^2]\omega^2 \dfrac{(\vec{\xi} \cdot \hat{\mathbf{e}}_b)}{c_s^2} = k^2 c_a^2 (\vec{\mathbf{k}} \cdot \hat{\mathbf{e}}_b)^2 (\vec{\xi} \cdot \hat{\mathbf{e}}_b)$. The dispersion relation for the magnetosonic waves is $\omega^4 - k^2(c_s^2 + c_a^2) = k^4 c_a^2 c_s^2 \cos^2\theta$.

Dynamo theory

MHD dynamo: mechanism by which a primary field $\vec{\mathbf{B}}_0(\vec{\mathbf{r}})$ is supported and amplified by mechanical motions of ionized electro-conducting gas or fluid. Current models require a seed magnetic field. Most models are kinematic: they assume a given velocity field of the fluid neglecting any back effect of the generated magnetic field on the fluid motions.

Equations involved: non-ideal ($\eta \neq 0$) astrophysical plasma MHD; the kinematic approx. with $\vec{\mathbf{v}}$, η fixed, bounded volume and initial magnetic energy.

Order of magnitude differential rotation dynamo, Elsasser units: the Euler equation (negligible viscosity) for a rotating fluid ($v \sim \Omega R$, order of magnitude) is $\dfrac{Dv}{Dt} \sim F + 2\Omega v + \Omega^2 R + \dfrac{1}{\rho} JB$. Consider the time average in a rotation period (lhs = 0). The gravitational and centrifugal forces don't contribute in a closed loop. Then, $JB \sim \rho \Omega v$ (ignoring direction). Ohm's law $\implies J \sim \sigma v B$. Combining both equations, we get $B \sim \sqrt{\dfrac{\rho\Omega}{\sigma}}$, which gives a scaling for the magnetic field produced by a differential rotation dynamo.

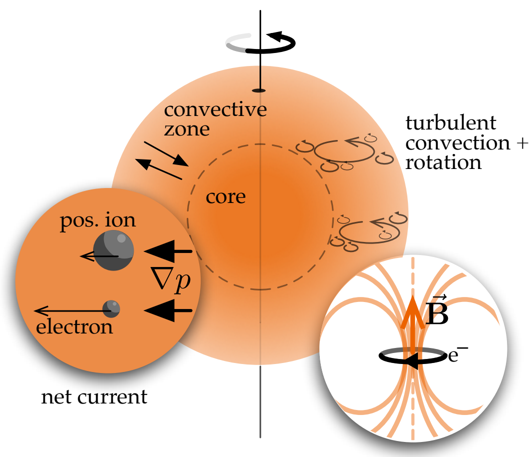
Turbulent dynamos: If there is strong turbulent motion, the *mean field theory* approach is needed, which relates the time evolution of the mean magnetic field to the statistical properties of the turbulent velocity field (uses a quasi-linear first-order approximation of average fields). This procedure defines a parameter α that measures kinetic helicity ($\mathcal{U}[\alpha] = \mathcal{U}[v]$). Convective cells in the interior of a star provide turbulent helical currents required for part of the dynamo.

Antidynamo theorems: there are several theorems that restrict the magnetic field produced by a dynamo. One of those theorems states that no axisymmetric magnetic field can be generated by an axially-symmetric current. This means that, although the magnetic field of the Sun and Earth have a strong dipole component, asymmetry is required for long-term stability. The magnetic field is then decomposed in a dipole-like component (the *poloidal field*) and a toroidal component in the direction of rotation.

$\alpha\Omega$-dynamos: A poloidal field is generated by the Coriolis acceleration + convection (see "Stellar dynamo"). From an initially poloidal field, differential rotation generates a toroidal field, since by leaving parts of the field "behind", it "twists" the magnetic field lines in the direction of rotation (Ω-effect). Now, the toroidal field is disrupted by the convection flow (due to flux freezing), ultimately deforming and twisting the toroidal field in small poloidal loops (after magnetic reconnection) (α-effect).

Stellar dynamo

Since plasma is quasi-neutral, some driving force must generate the magnetic field. Part of the dynamo problem is to find a flow $\vec{\mathbf{v}}$ such that, when inserted to the induction equation, leads to amplification of $\vec{\mathbf{B}}$.



The following is a very simplified model of a solar dynamo, which is a $\alpha\Omega$-dynamo:

1. *Convection*: it takes place in the convective zone of a star, and occurs when the plasma is heated, so it expands and becomes buoyant (difference in pressure drives a force).

2. *Net current*: when the difference in pressure arises, the Euler equation predicts there will be different acceleration for electrons and positive ions because of the difference in mass (density). This creates a net current.

3. *Rotation and turbulence*: rotation makes current loops in the convection flow, and turbulence makes lots of small loops. Those loops of net current generate the *poloidal* magnetic field, that is, a dipole field made from the sum of all the loops.

4. *Problems with the model*: this model takes into account the primary contributors (energetically speaking) to $\vec{\mathbf{v}}$, but ignores the resistance of the flow, which makes the dynamo not self-sustainable. Also, the magnetic field of the Sun is not only poloidal, but it has other components and it is highly dynamic. Those other components arise from differential rotation, and eventually, also contribute to the self-sustainability of the poloidal field. This process doesn't explain the reversal in polarity observed periodically in the Sun. The loops actually look more like helices thanks to the Coriolis acceleration (convection is spherically radial, so it has a vertical component). As we can see, the dynamo problem is inherently nonlinear.

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